

The quantum SKdV_{1,4} equation at $c = 3$

P. Mathieu¹

Département de Physique, Université Laval, Québec, Canada G1K 7P4

Abstract

At $c = 3$, two of the three integrable quantum $N = 2$ supersymmetric Korteweg-de Vries equations become identical (SKdV₁ and SKdV₄). Quite remarkably, all their conservation laws can be written in closed form, which provides thus a simple constructive integrability proof.

¹ Work supported by NSERC (Canada)

The quantum extension of the usual Korteweg-de Vries (KdV) equation is rather easily formulated [1, 2]: it is the canonical equation obtained from the quantum version of the KdV second Hamiltonian structure (i.e., the energy-momentum tensor OPE) and the normal ordered form of the corresponding KdV Hamiltonian:

$$\dot{T} = -[H, T], \quad H = \frac{1}{2\pi i} \oint dz (TT)$$

Since the classical Hamiltonian has a single term, its quantum form is not ambiguous. The quantum extension of the supersymmetric KdV equation is also defined unambiguously [3, 4]. This is no longer true for the three integrable $N = 2$ supersymmetric KdV (SKdV) equations [5] since the three different Hamiltonian contains two terms, hence a troublesome relative coefficient whose quantum form cannot be fixed a priori. More explicitly, this classical Hamiltonian is [6]:

$$H_{\alpha}^{\text{class}} = \int dX \left\{ \mathcal{U}^3 - \frac{3}{2\alpha} (\mathcal{U}[D^+, D^-]\mathcal{U}) \right\}$$

where $\mathcal{U}(x, \theta^+, \theta^-, t)$ is the classical form of the $N = 2$ superenergy-momentum tensor, $dX = dx d\theta^+ d\theta^-$ and $D^{\pm} = \partial_{\theta^{\mp}} + \theta^{\pm} \partial_x$. The corresponding KdV equation is the canonical equation obtained from this Hamiltonian and the Poisson bracket version of the $N = 2$ superconformal algebra. It is integrable for only three values of α , namely 1, -2, 4 [6, 7].

An elegant way of determining the quantum form of these integrable equations is based on their correspondence with the (conjectured) integrable perturbations of the $N = 2$ minimal models with $c = 3K/(K + 2)$ [5, 8, 9]. This fixes the relative coefficient of the model defining quantum Hamiltonian. Identifying the equations by the corresponding perturbation (that is, the chiral field label ℓ which represents the perturbation Φ^{ℓ}) as well as with the corresponding value of the classical label α , we have [5]²

$$\begin{aligned} \ell = 1 \quad H_{\alpha=4} &= \oint dZ \left\{ (\tilde{\mathbf{T}}(\tilde{\mathbf{T}}\tilde{\mathbf{T}})) + \frac{1}{16} (c - 3) (\tilde{\mathbf{T}}[D^+, D^-]\tilde{\mathbf{T}}) \right\} \\ \ell = 2 \quad H_{\alpha=1} &= \oint dZ \left\{ (\tilde{\mathbf{T}}(\tilde{\mathbf{T}}\tilde{\mathbf{T}})) + \frac{1}{4} (c - 3) (\tilde{\mathbf{T}}[D^+, D^-]\tilde{\mathbf{T}}) \right\} \\ \ell = K \quad H_{\alpha=-2} &= \oint dZ \left\{ (\tilde{\mathbf{T}}(\tilde{\mathbf{T}}\tilde{\mathbf{T}})) - \frac{1}{8} (c - 12) (\tilde{\mathbf{T}}[D^+, D^-]\tilde{\mathbf{T}}) \right\} \end{aligned}$$

² Further support for the conjectured integrability of the $\ell = 1$ and $\ell = K$ perturbations is presented in [10] and in [11] respectively. Moreover, for each value of K , the perturbations $\ell = 1, 2$ have been related to quantum affine Toda theories at a particular value of the coupling [9] and these Toda models have been proved to be integrable in [12].

where $\tilde{\mathbf{T}}(Z)$ is the $N = 2$ superenergy-momentum tensor

$$\tilde{\mathbf{T}}(Z) = J(z) + \frac{1}{2}\theta^- G^+(z) - \frac{1}{2}\theta^+ G^-(z) + \theta^+ \theta^- T(z)$$

whose OPE reads

$$\tilde{\mathbf{T}}(Z_1) \tilde{\mathbf{T}}(Z_2) = \frac{c/12}{Z_{12}^2} + \frac{\theta_{12}^+ \theta_{12}^- \tilde{\mathbf{T}}(Z_2)}{Z_{12}^2} + \frac{\theta_{12}^+ D^- \tilde{\mathbf{T}}(Z_2)}{2Z_{12}} - \frac{\theta_{12}^- D^+ \tilde{\mathbf{T}}(Z_2)}{2Z_{12}} + \frac{\theta_{12}^+ \theta_{12}^- \partial \tilde{\mathbf{T}}(Z_2)}{Z_{12}}$$

with $Z_{12} \equiv z_1 - z_2 - \theta_1^+ \theta_2^- - \theta_1^- \theta_2^+$.

In the classical limit ($c \rightarrow \pm \infty$), the ratios of the three relative coefficients are seen to be the same as the inverse ratios of the quoted values of α and the classical Hamiltonians are recovered by the relation $\tilde{\mathbf{T}} = -c\mathcal{U}/6$.

Denote by qSKdV_α the equations obtained canonically from the quantum Hamiltonian H_α . Since at $c = 3$, $H_{\alpha=4} = H_{\alpha=1}$ ³, the two equations qSKdV_1 and qSKdV_4 merge into a single one.⁴ It turns out that their conserved densities have an extremely simple form, namely

$$(\tilde{\mathbf{T}}^n) = (\cdots (((\tilde{\mathbf{T}}\tilde{\mathbf{T}})\tilde{\mathbf{T}})\tilde{\mathbf{T}}) \cdots \tilde{\mathbf{T}}) \quad (n \text{ factors}) \quad (1)$$

Hence, every conserved density has a single term but normally ordered toward the left. This is the exact analog of the qKdV conservation laws at $c = -2$ [2, 13]. The rest of this note is devoted to the proof of this result, which boils down to a simple exercise in normal ordering rearrangements.

The idea of the proof is to use the $c = 3$ free field representation [14] :

$$\tilde{\mathbf{T}} = -\frac{1}{4}(D^+ S_+ D^- S_-)$$

³ By treating this value of c as the limiting minimal model with $K \rightarrow \infty$, we further verify that the degenerate equation of the perturbing field Φ^2 is a descendant of that of Φ (both perturbations have vanishing conformal dimension at $c = 3$).

⁴ Similarly, when $c = 6$, $H_{\alpha=1} = H_{\alpha=-2}$ and for $c = 9$, $H_{\alpha=4} = H_{\alpha=-2}$. Moreover, when $c = 1$ or $3/2$, thanks to the vacuum singular vector:

$$\begin{aligned} c = 1 : & \quad \left(J_{-1}^2 - \frac{1}{6} L_{-2} \right) |0\rangle \\ c = \frac{3}{2} : & \quad \left(J_{-1}^3 - \frac{3}{8} J_{-1} L_{-2} - \frac{1}{16} J_{-3} - \frac{3}{64} L_{-3} + \frac{3}{64} G_{-3/2}^+ G_{-3/2}^- \right) |0\rangle \end{aligned}$$

the three Hamiltonians reduce to $\oint dZ (\tilde{\mathbf{T}}(\tilde{\mathbf{T}}\tilde{\mathbf{T}}))$.

As usual parentheses denote normal ordering. The free field OPE's are

$$S_+(Z_1) S_-(Z_2) \sim -\ln Z_{12} + \frac{\theta_{12}^+ \theta_{12}^-}{Z_{12}}$$

and $S_+ S_+ \sim S_- S_- \sim 0$. S_+ and S_- are chiral primary fields: $D^- S_+ = D^+ S_- = 0$. The canonical equations of these fields take an extremely simple form. As a result, the explicit expression for all the conserved charges of this chiral free field system can be written down readily. The final step amounts to prove that these can be reexpressed in terms of $\tilde{\mathbf{T}}$, according to (1).

At $c = 3$, the Hamiltonian for the model under consideration is

$$\tilde{H}_3 = \oint dZ (\tilde{\mathbf{T}}(\tilde{\mathbf{T}}\tilde{\mathbf{T}})) = -\frac{3}{8} \oint dZ (S_+^{(3)} S_-)$$

where $S_+^{(n)} = \partial^n S_+$. The canonical equations for the fields S_+, S_-

$$\dot{S}_+ = -[\tilde{H}, S_+], \quad \dot{S}_- = -[\tilde{H}, S_-]$$

reduce to (with an appropriate time rescaling):

$$\dot{S}_+ = S_+^{(3)}, \quad \dot{S}_- = S_-^{(3)}$$

The infinite sequence of conserved charge for this system reads then

$$\tilde{H}_n = \oint dZ (S_+^{(n)} S_-)$$

where n is any positive integer. It is simple to verify that $[\tilde{H}_n, \tilde{H}_m] = 0$. Now, as for the KdV equation at $c = -2$, these charges can be written in terms of the normal ordered powers of $\tilde{\mathbf{T}}$, but with a left nesting. This is the announced result:

$$\tilde{H}_n = \oint dZ (\tilde{\mathbf{T}}^n)$$

which will be proved by recursion.

Normal ordering of $N = 2$ superfield is defined as

$$(\mathcal{A}\mathcal{B})(Z_2) = \frac{1}{2\pi i} \oint dZ_1 \frac{\theta_{12}^+ \theta_{12}^-}{Z_{12}} \mathcal{A}(Z_1) \mathcal{B}(Z_2)$$

Note in particular that if

$$\mathcal{A}(Z_1) \mathcal{B}(Z_2) = \sum_{n=N}^{-\infty} Z_{12}^{-n} [\mathcal{C}_n(Z_2) + \theta_{12}^+ \mathcal{D}_n(Z_2) + \theta_{12}^- \mathcal{E}_n(Z_2) + \theta_{12}^+ \theta_{12}^- \mathcal{F}_n(Z_2)]$$

the normal ordered commutator (anticommutator if \mathcal{A} and \mathcal{B} are both fermionic) is

$$([\mathcal{A}, \mathcal{B}](Z_2) = \sum_{n>0} \frac{(-1)^n}{n!} \partial^n \mathcal{C}_n(Z_2)$$

Standard reordering manipulations rely on the rearrangement lemmas of [15], e.g.,

$$((AB)(CD)) = (C(D(AB))) + (((AB), C])D) + (C([(AB), D]))$$

To simplify somewhat the notation, we will set

$$\Lambda_{\pm} \equiv D^{\pm} S_{\pm}$$

We easily find that

$$\begin{aligned} \overleftarrow{(\tilde{\mathbf{T}}^2)} &= (\tilde{\mathbf{T}}\tilde{\mathbf{T}}) = -\frac{1}{8} \left[(\Lambda_+^{(1)} \Lambda_-) - (\Lambda_+ \Lambda_-^{(1)}) \right] \\ \overleftarrow{(\tilde{\mathbf{T}}^3)} &= ((\tilde{\mathbf{T}}\tilde{\mathbf{T}})\tilde{\mathbf{T}}) = -\frac{3}{32} \left[(\Lambda_+^{(2)} \Lambda_-) + (\Lambda_+ \Lambda_-^{(2)}) \right] \end{aligned}$$

Let us then assume that $\overleftarrow{(\tilde{\mathbf{T}}^n)}$ has the form

$$\overleftarrow{(\tilde{\mathbf{T}}^n)} = b_n \left[(\Lambda_+^{(n-1)} \Lambda_-) + (-1)^{n-1} (\Lambda_+ \Lambda_-^{(n-1)}) \right] \quad (2)$$

We now prove that

$$\overleftarrow{(\tilde{\mathbf{T}}^{n+1})} = ((\overleftarrow{(\tilde{\mathbf{T}}^n)}\tilde{\mathbf{T}}) = \left(\frac{n+1}{2n} \right) b_n \left[(\Lambda_+^{(n)} \Lambda_-) + (-1)^n (\Lambda_+ \Lambda_-^{(n)}) \right] \quad (3)$$

The proof can be worked out in few lines. $\overleftarrow{(\tilde{\mathbf{T}}^{n+1})}$ is equal to

$$\overleftarrow{(\tilde{\mathbf{T}}^{n+1})} = -\frac{b_n}{4} [\Gamma_1 + (-1)^{n-1} \Gamma_2] \quad (4)$$

where

$$\begin{aligned}
\Gamma_1 &= \left((\Lambda_+^{(n-1)} \Lambda_-) (\Lambda_+ \Lambda_-) \right) \\
&= \left(([\Lambda_+^{(n-1)} \Lambda_-, \Lambda_+]) \Lambda_- \right) + \left(\Lambda_+ ([\Lambda_+^{(n-1)} \Lambda_-, \Lambda_-]) \right) \\
&= -2 (\Lambda_+^{(n)} \Lambda_-) + (-1)^{n+1} \frac{2}{n} (\Lambda_+ \Lambda_-^{(n)}) \\
\Gamma_2 &= \left((\Lambda_+ \Lambda_-^{(n-1)}) (\Lambda_+ \Lambda_-) \right) \\
&= \left(([\Lambda_+ \Lambda_-^{(n-1)}, \Lambda_+]) \Lambda_- \right) + \left(\Lambda_+ ([\Lambda_+ \Lambda_-^{(n-1)}, \Lambda_-]) \right) \\
&= (-1)^n \frac{2}{n} (\Lambda_+^{(n)} \Lambda_-) + 2 (\Lambda_+ \Lambda_-^{(n)})
\end{aligned} \tag{5}$$

The substitution of (5) into (4) yields (3). The recursion argument proves (2) and fixes the coefficient b_n :

$$b_n = -\frac{n}{2^{n+2}}$$

Notice that the quartic terms disappear in $(\tilde{\mathbf{T}}^{n+1})$ thanks to the fermionic character of the Λ_{\pm} . For this it is crucial that $(\tilde{\mathbf{T}}^n)$ be composed of terms which all contain a factor Λ_+ or Λ_- without derivatives. This property is lost when the powers of $\tilde{\mathbf{T}}$ are ordered toward the right (and as a result, the conserved densities no longer have a simple form).

Up to a total derivative, $(\tilde{\mathbf{T}}^n)$ is proportional to $(S_+^{(n)} S_-)$. As already indicated, the integrals \tilde{H}_n are mutually commuting. We have thus obtain a rigorous and constructive integrability proof for the qSKdV_{1,4} equation at $c = 3$.

For the other SKdV equation, the presence of the term $(\tilde{\mathbf{T}}[D^+, D^-] \tilde{\mathbf{T}})$ in the Hamiltonian induces quartic contribution in the fields S_+, S_- that generates cubic terms in the equations for S_+ and S_- . These terms couple the two fields and the equation become sufficiently complicated to prevent a closed form expression for their conservation laws.

1. B.A. Kuperschmidt and P. Mathieu, Phys. Lett. **B227** (1989) 245.
2. R. Sasaki and I. Yamanaka, Adv. Stud. in Pure Math. **16** (1988) 271.
3. P. Mathieu, Nucl. Phys. **B336** (1990) 338.
4. I. Yamanaka and R. Sasaki, Prog. Theor. Phys. **79** (1988) 1167.
5. P. Mathieu and M.A. Walton, Phys. Lett. **B254** (1991) 106.
6. C.-A. Laberge and P. Mathieu, Phys. Lett. **B215** (1988) 718; P. Labelle and P. Mathieu, J. Math. Phys. **32** (1991) 923.

7. Z. Popowicz, Phys. Lett. **A174** (1994) 411.
8. P. Fendley, S.D. Mathur, C. Vafa and N. Warner, Phys. Lett. **B243** (1990) 257.
9. P. Fendley, W. Lerche, S.D. Mathur and N. P. Warner, Nucl. Phys. **B348** (1991) 66.
10. P. Di Francesco and P. Mathieu, Phys. Lett. **B278** (1992) 79.
11. T. Eguchi and S.K. Yang, Mod. Phys. Lett. **A5** (1990) 1693.
12. B. Feigin and E. Frenkel, *Integrals of motion and quantum groups*, proceedings of C.I.M.E. Summer School on ‘Integrable systems and Quantum groups’, 1993 hep-th/9310022.
13. P. Di Francesco, P. Mathieu and D. Sénéchal, Mod. Phys. Lett. **A7** (1992) 701.
14. G. Mussardo, G. Sotkov and M. Stanishkov, Int. J. Mod. Phys. **A4** (1989) 1135.
15. F.A. Bais, P. Bouwknegt, K. Schoutens and M. Surridge, Nucl. Phys. **B304** (1988) 348.